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SUBJECT GTARG ERROR MODELS

GTARG is the TOPEX/POSEIDON ground track maintenance maneuver targeting program. It combines orbit prediction and targeting algorithms to design maneuvers which ensure that the ground track is maintained within a ±1 km. wide control band about an ≈9.9 day repeat pattern. Error models include the uncertainties due to orbit determination, maneuver execution errrors, drag unpredictability, and the knowledge of unspecified along-track satellite-fixed forces ("boost/decay" forces). These error models are used to define an envelope of uncertainty about the predicted ground track with a desired confidence level, usually 95%.

This memorandum summarizes the error models as they are implemented in GTARG.

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Definition of the Error Envelope

GTARG calculates the error envelope as

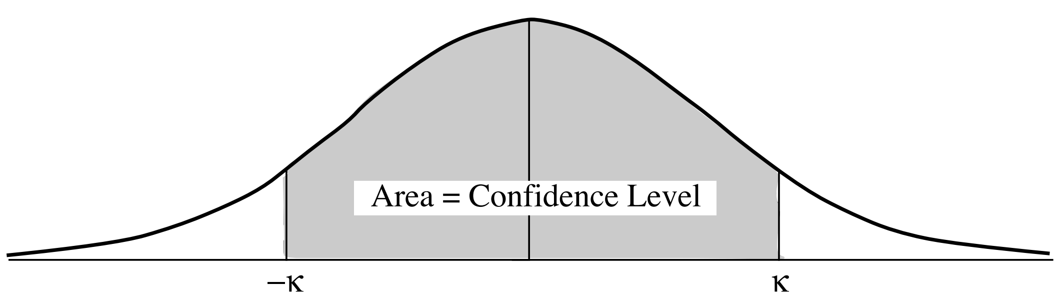
$$\sigma_{\Delta\lambda}(t) = \sqrt{\kappa^2_{\Delta V,OD}\left\{\left[\sigma_{\Delta\lambda,\Delta V}(t)\right]^2 + \left[\sigma_{\Delta\lambda,OD}(t)\right]^2\right\} + \kappa^2_{Drag}\left[\sigma_{\Delta\lambda,Drag}(t)\right]^2 + \kappa^2_{da/dt}\left[\sigma_{\Delta\lambda,da/dt}(t)\right]^2} \qquad (1)$$

where

$\sigma_{\Delta\lambda,\Delta V}$ = 1-σ uncertainty in ground track Δλ due to maneuver execution errors;
$\sigma_{\Delta\lambda,OD}$ = 1-σ uncertainty in ground track Δλ due to orbit determination errors;
$\sigma_{\Delta\lambda,Drag}$ = 1-σ uncertainty in ground track Δλ due to drag prediction errors;
$\sigma_{\Delta\lambda,da/dt}$ = 1-σ uncertainty in ground track Δλ due to "boost" prediction errors;
$\kappa_{\Delta V,OD}$ = scaling factor for maneuver execution and orbit determination errors;
κ_{Drag} = scaling factor for drag error;
$\kappa_{da/dt}$ = scaling factor for "boost/decay" error.

The validity of equation (1) is contingent upon the assumptions that the four error sources are described by uncorrelated random variables. An assumption of independence between the error sources is sufficient to prove that the variables are uncorrelated. The scaling factors κ give contribution to the eror envelope of each error source in standard deviations. The scale factors can be related to the confidence level by assuming that the error sources are distributed as random variables with a specified probability distribution. Figure 1 shows the relationship between the confidence level and κ by assuming a normal distribution. Values of κ for various levels of confidence are shown in figures 2 and 3.

Figure 1.
Definition of confidence levels for error sources which are represented as random variables with a standard normal distiribution. The confidence level A and scale factor k are related by $A = \frac{1}{\sqrt{2\pi}}\int_{-\kappa}^{\kappa} e^{-z^2/2}\,dz$.



Input to GTARG is provided in the form of 1-σ uncertainties in the natural units of the error source, e.g.,

Δa_{OD} = uncertainty in semi-major axis due to orbit determination (1-σ);
$\delta\Delta V_{fixed}$ = uncertainty in maneuver magnitude due to fixed error (1-σ);

$\delta\Delta V_{proportional} / \Delta V$ = fractional uncertainty in maneuver magnitude due to proportional error (1-σ);

$\Delta F_{10.7}$ = uncertainty in solar flux (1-σ);

$\Delta \overline{F_{10.7}}$ = uncertainty in 81-day mean solar flux (1-σ);

ΔK_p = uncertainty in geomagnetic index;

$\Delta\left(\frac{da}{dt}\right)$ = uncertainty in da/dt due to "boost/decay" errors.

Figure 2.
Relationship between scale factor and confidence level.

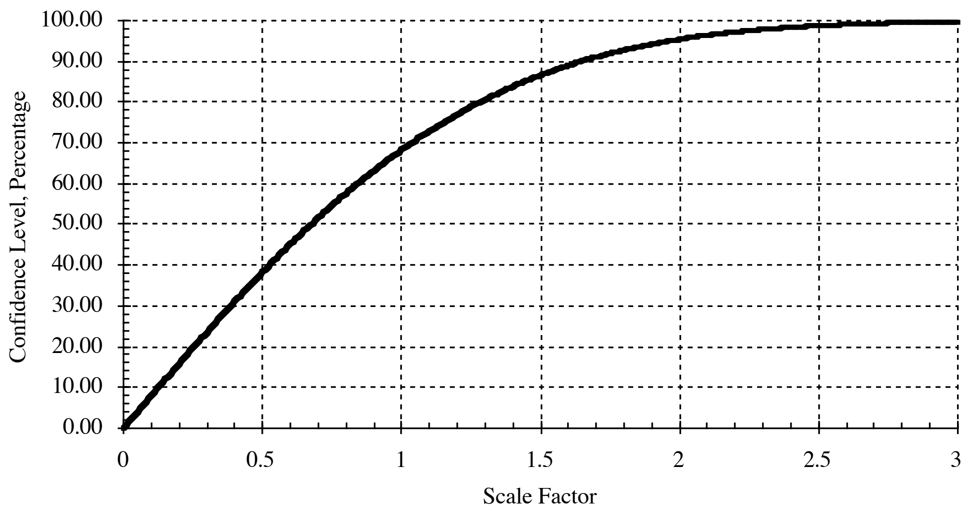
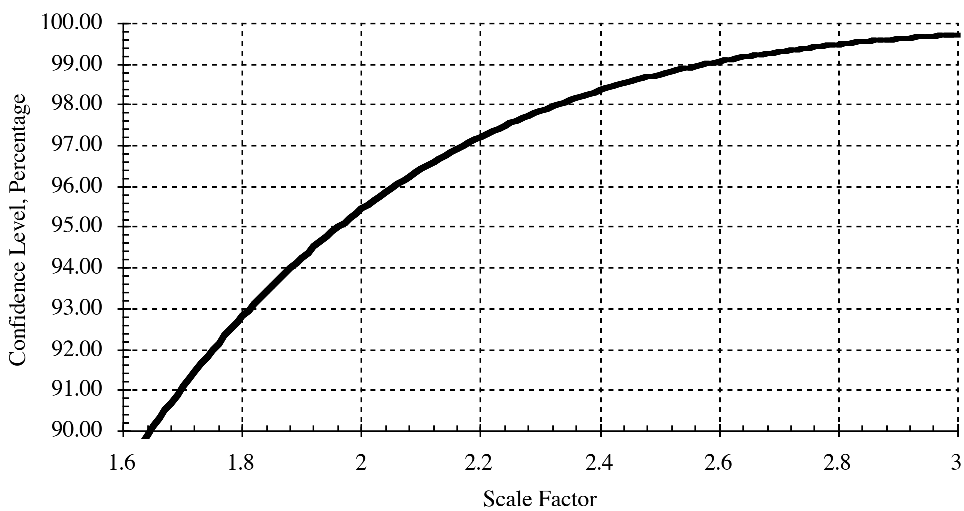


Figure 3
Relationship between scale factor and confidence level for high confidence levels.



GTARG has the task of converting the errors from their natural units into ground track units, propagating them , and combining them together at a later time to determine the width of the error envelope.

The following sections summarize the models which are used for each of the four error sources in GTARG. While two of these (Maneuver implementation uncertainty and orbit determination uncertainty) have been described previously,[1,2] they are included here for completeness.

Maneuver Implementation Uncertainty

The fixed and proportional errors are treated as independent random variables and added in quadrature to produce a total maneuver implementation uncertainty[1],

$$\delta\Delta V = \sqrt{\delta\Delta V_{fixed}^2 + \delta\Delta V_{proportional}^2} \tag{2}$$

While it would be legitimate to also include pointing errors in equation (2), they produce only very small errors. Typical maneuver magnitudes range from 1 to 10 mm/sec. The pointing error contributes $\delta\Delta v_{pointing} / \Delta v \approx (1 - \cos\delta\theta) < 0.00035$ for $\delta\theta<1.5°$, and <0.0014 for $\delta\theta<3°$. Hence these errors are small enough to be ignored in GTARG.

The total maneuver uncertainty is related to the ground track uncertainty at a later time t by[1,2]

$$\frac{\partial\Delta\lambda}{\partial\Delta V} = -\frac{3\omega_e t}{V} \cong -16.96\, t(days) \frac{meters}{mm / \sec} \tag{3}$$

and hence the contribution of maneuver uncertainty to equation (1) is approximated as

$$\sigma_{\Delta\lambda,\Delta V}(t) \cong \left(\frac{\partial\Delta\lambda}{\partial\Delta V}\right)\delta\Delta V \cong -\frac{3\omega_e t\, \delta\Delta V}{V} \tag{4}$$

Orbit Determination Uncertainty

Orbit determination uncertainty is expressed as an error in the semi-major axis, as this has the dominant effect on the ground track. The relationship between the two is[1,2]

$$\frac{\partial\Delta\lambda}{\partial a} \cong \frac{3}{2}\frac{\omega_e}{a} t \cong 7.81\, t(days) \frac{meters}{meter} \tag{5}$$

and hence

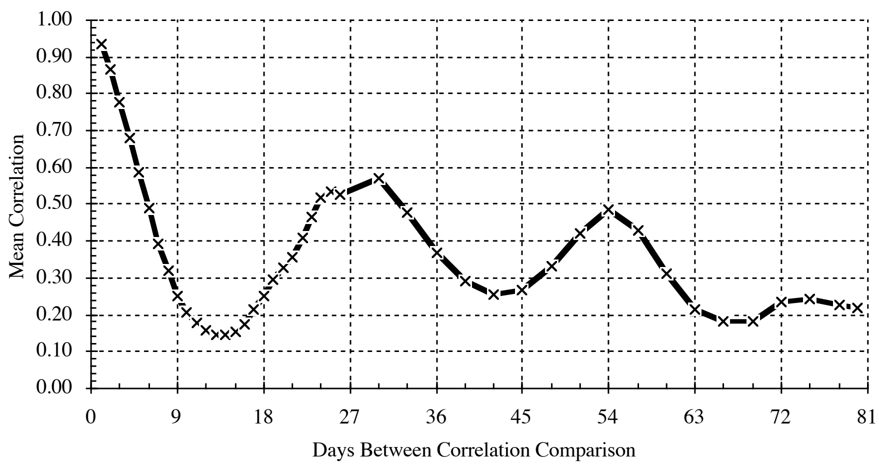
$$\sigma_{\Delta\lambda,OD} \cong \left(\frac{\partial\Delta\lambda}{\partial a}\right)\Delta a_{OD} \cong \frac{3\omega_e t \Delta a_{OD}}{2a} \tag{6}$$

Accumulated Errors

The error sources discussed so far, orbit determination uncertainty and maneuver execution error, are simple error sources. They occur once, and are propagated as algebraic functions of time. The remaining error sources, drag and "boost/decay" prediction uncertainties, occur at each propagation step. Successive errors must be combined and propagated forward to the subsequent step.

The manner of combining these errors depend on what assumptions are made for the correlation between the uncertainty predictions on different days. For example, the behavior of the "boost/decay" error appears to be well described, and the daily uncertainties are nearly independent and hence can be treated as uncorrelated random variables. The daily solar flux predictions, however, show high correlation, as illustrated in figure 4.

Figure 4.
Mean correlation between solar flux prediction error and the solar flux prediction error N days later. The results are based upon the SESC 27-Day outlooks, repeated three times, for 1992 (53 prediction sets). The 27-day periodicity is due to the solar rotation period. The actual correlation (not the absolute value) is shown.



If X_1 and X_2 are two random variables with standard deviations σ_1 and σ_2, and means μ_1 and μ_2, then the standard deviation of the random variable $Y = X_1 + X_2$ is given by

$$\sigma_Y^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2 \tag{7}$$

where ρ is the correlation between X_1 and X_2, defined as

$$\rho = \frac{1}{\sigma_1\sigma_2} E\left[(X_1 - \mu_1)(X_2 - \mu_2)\right] \tag{7a}$$

and $|\rho| \leq 1$. The operator E(x) gives the expected value of the argument x. The general result for the sum $Y = \sum_{i=1}^{n} X_i$ of n random variables X_i with standard deviations σ_i is

$$\sigma_Y^2 = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{j=2}^{n}\sum_{i=1}^{j} \rho_{ij}\sigma_i\sigma_j \tag{8}$$

where ρ_{ij} is the correlation between X_i and X_j (this is Theorem 5.4.1 of reference [3]). Let X_i be the random variable describing the error introduced on orbit *i* and X_j the random variable describing the error introduced on orbit *j*, where $j>i$. Then Y is the random variable describing the total error accumulated and propagated from all earlier days on day *j*. The total standard deviation σ_Y given by equation (8) is the 1-σ error envelope due to the total contribution of all previous days errors on day *j*.

Two extremes described by equation (7) correspond to $\rho = 0$ and $\rho = 1$,

$$\sigma^2 = \begin{cases} \sigma_1^2 + \sigma_2^2, & \rho = 0 \ (\textit{optimistic case}) \\ (\sigma_1 + \sigma_2)^2, & \rho = 1 \ (\textit{pessimistic case}) \end{cases} \tag{9}$$

Errors add linearly in the pessimistic case, and in quadrature in the optimistic case.

The general case (equation 8) as it applies to drag and "boost/decay" uncertainty has not been implemented in GTARG due to the computational complexity it would entail. However, both extreme cases (optimistic and pessimistic) have been implemented.

Drag Prediction Uncertainty

At each propagation step, three densities are calculated,

$$\rho_0(t) = \rho\left[F_{10.7}(t), \overline{F_{10.7}}(t), K_p(t)\right] = \text{nominal density at t;}$$
$$\rho_{Hi}(t) = \rho\left[F_{10.7}(t) + \Delta F_{10.7}(t), \overline{F_{10.7}}(t) + \Delta\overline{F_{10.7}}(t), K_p(t) + \Delta K_p(t)\right] = \text{high density at t;}$$
$$\rho_{Lo}(t) = \rho\left[F_{10.7}(t) - \Delta F_{10.7}(t), \overline{F_{10.7}}(t) - \Delta\overline{F_{10.7}}(t), K_p(t) - \Delta K_p(t)\right] = \text{low density at t.}$$

From the three density functions, three ground tracks are propagated simultaneously:

$\Delta\lambda_{Nom}(t)$ = ground track based on $\rho_0(t)$.
$\Delta\lambda_{Hi}(t)$ = ground track based on $\rho_{Hi}(t)$.
$\Delta\lambda_{Lo}(t)$ = ground track based on $\rho_{Lo}(t)$.

and the following differences are calculated

$$\delta_{East} = \Delta\lambda_{Hi} - \Delta\lambda_{Nom} \tag{10a}$$
$$\delta_{West} = \Delta\lambda_{Nom} - \Delta\lambda_{Lo} \tag{10b}$$

At this point, a choice of two error models are available: (a)an optimistic error model, in which the error sources at time step t+Δt are assumed to be completely independent of the error sources at time t, and (b) a pessimistic model in which the error sources are completely dependent.

Optimistic Drag Error Model. Consider the high and low drag errors separately. Let $\delta_i = \delta(t_i)$ be the difference calculated by either of equations (10) at time t_i, and let $\sigma_i = \sigma_{\Delta\lambda, Drag}(t)$ be the accrued ground track error at that time due to the drag uncertainty. Then the errors at each time step are assumed to add in quadrature with the accrued error from the previous time step:

$$\sigma_1 = 0 \tag{11a}$$

$$\sigma_{i+1} = \sqrt{(\delta_{i+1} - \delta_i)^2 + \sigma_i^2} \tag{11b}$$

Pessimistic Drag Error Model. Again consider the high and low drag errors separately. Let $\delta_i = \delta(t_i)$ be the difference calculated by either of equations (10) at time t_i, and let $\sigma_i = \sigma_{\Delta\lambda, Drag}(t)$ be the accrued ground track error at that time due to the drag uncertainty. Then the errors at each time step are assumed to add linearly with the accrued error from the previous time step:

$$\sigma_1 = 0 \tag{12a}$$

$$\sigma_{i+1} = \delta_{i+1} - \delta_i + \sigma_i = \delta_{i+1} \tag{12b}$$

"Boost/Decay" Prediction Uncertainty.

GTARG implements four different error models for the "boost/decay" prediction uncertainy. In order of complexity, they are

(a) Constant error pessimisitic model. An error of fixed magnitude is generated every day, which is assumed to be completely dependent on the previous day's error. The ground track error is accrued linearly.

(b) Constant error optimistic model. An error of fixed magnitude is generated every day, which is assumed to be completely independent of the previous day's error. The ground track error is accrued in quadrature.

(c) Variable error pessimistic model. A time varying error occurs, which is completely dependent on the previous daily error. The ground track error is accrued linearly.

(d) Variable error optimistic model. A time varying error occurs, which is completely independent on the previous daily error. The ground track error is accrued in quadrature.

Each of the four models is described separately.

Constant error pessimisitic model.

A fixed error Δa is assumed to occur each orbit. The ground track error due to the Δa from a single orbit will grow in the same manner as if it were orbit determination error, as described by equation (6). Ignoring the change in the period P, after the N^{th} orbit, a time $t = NP$ has elapsed, and the total ground track error is just the sum of the propagated errors introduced originally during each of the earlier N orbits. Then

$$\begin{aligned}\sigma_{\Delta\lambda,da/dt}(t) &= \sum_{i=1}^{N-1}\frac{3}{2}\frac{\omega_e\Delta a}{a}iP \\ &= \frac{3}{2}\frac{\omega_e\Delta a}{a}\left[\frac{N(N-1)}{2}\right]P \\ &= \frac{3}{4}\frac{\omega_e\Delta a}{a}\left[t\left(\frac{t}{P}-1\right)\right] \\ &\approx 3.905\Delta a\left(\frac{meters}{rev}\right)\left[t(days)\left[\frac{t(days)}{P(days)}-1\right]\right]\end{aligned} \tag{13}$$

Constant error optimistic model.

Rather than adding the errors linearly, as in equation (13), they are added in quadrature. After N orbits, $t \equiv t_N \cong NP$ and

$$\sigma^2_{\Delta\lambda,da/dt}(t_N) = \left(\frac{3}{2}\frac{\omega_e}{a}\Delta a\right)^2\sum_{i=1}^{N-1}(t_N - t_i)^2 = K^2(\Delta a)^2\sum_{i=1}^{N-1}(t_N - t_i)^2 \tag{14}$$

where $K = 3\omega_e/2a$. Again, ignoring the change in period, $t_i = iP$ and hence

$$\begin{aligned}\sigma^2_{\Delta\lambda,da/dt}(t_N) &= K^2(P\Delta a)^2\sum_{i=1}^{N-1}(N-i)^2 \\ &= K^2(P\Delta a)^2\sum_{i=1}^{N-1}i^2 \\ &= K^2(P\Delta a)^2\frac{(N-1)N(2N-1)}{6}\end{aligned} \tag{15}$$

The accrued error after a time $t=NP$ is then

$$\sigma^2_{\Delta\lambda,da/dt}(t_N) = \frac{3}{2}\frac{\omega_e\Delta a}{a}\sqrt{\frac{(t-P)t(2t-P)}{6P}} \tag{16}$$

Variable error pessimistic model.

Algorithm. Let the propagation step size be M orbits. Ignoring the change in period, the time after N orbits is $t = t_N = NP$. Use the notation $\sigma_N \equiv \sigma_{\Delta\lambda,da/dt}(t_N)$. Then the error envelope is calculated according to the following algorithm:

$$\boxed{\begin{aligned} K &= \frac{3}{2}\frac{\omega_e}{a} \\ \delta_1 &= M^2 PK\Delta a_1 \\ \delta_{N+M} &= \delta_N + M^2 PK\Delta a_N \\ \sigma_1 &= 0 \\ \sigma_{N+M} &= \sigma_N + \delta_N + KP\Delta a_N \left[\tfrac{1}{2}M(M-1)\right] \end{aligned}}$$

[Algorithm A]

<u>Derivation</u>. In this section, the notation σ_N is used to indicate the 1-σ uncertainty in the ground track (i.e., $\sigma_N \equiv \sigma_{\Delta\lambda, da/dt}(t_N)$ where $t = t_N = NP$).

In this model, an error Δa_i is introduced during the ith orbit. The error from orbit i propagates according to equation (6). The errors are assumed to add linearly. This model is more complex; rather than being described by a simple analytic function of t, recursion relations must be used. Let orbit *i* start at time t_i. At orbit *N*, the accrued error is

$$\sigma_N = K\sum_{i=1}^{N-1}\Delta a_i (t_N - t_i) \tag{17}$$

where $K = 3\omega_e / 2a$. One orbit later, at $t_{N+1} = t_N + P$,

$$\begin{aligned} \sigma_{N+1} &= K\sum_{i=1}^{N}\Delta a_i (t_N + P - t_i) \\ &= K\sum_{i=1}^{N-1}\Delta a_i (t_N + P - t_i) + K\Delta a_N (t_N + P - t_N) \\ &= K\sum_{i=1}^{N-1}\Delta a_i (t_N - t_i) + K\sum_{i=1}^{N-1}\Delta a_i P + K\Delta a_N P \\ &= \sigma_N + KP\sum_{i=1}^{N}\Delta a_i \\ &= \sigma_N + \delta_{N+1} \end{aligned} \tag{18}$$

In the final step of equation (18), the auxillary function δ has been defined, where

$$\delta_N = K\sum_{i=1}^{N-1}\Delta a_i = \delta_{N-1} + K\Delta a_{N-1} \tag{19}$$

Equations (18) and (19) can be combined into the following algorithm:

$$\boxed{\begin{aligned} K &= \frac{3}{2}\frac{\omega_e}{a} \\ \delta_1 &= 0 \\ \delta_i &= \delta_{i-1} + KP\Delta a_{i-1} \\ \sigma_1 &= 0 \\ \sigma_i &= \Delta\lambda_{i-1} + \delta_i \end{aligned}} \tag{20}$$

The algorithm expressed by equation (20) is not sufficient for implemention unless the propagation step size is precisely one orbit. GTARG allows an integration step size of M orbits. Equation (20) must be modified as follows. After N+M orbits, using $t_i = iP$

$$\begin{aligned}
\sigma_{N+M} &= K \sum_{i=1}^{N+M-1} \Delta a_i (N+M-i)P \\
&= K \sum_{i=1}^{N-1} \Delta a_i (N+M-i)P + K\Delta a_N P \sum_{i=N}^{N+M-1} (N+M-i) \\
&= K \sum_{i=1}^{N-1} \Delta a_i (N-i)P + K \sum_{i=1}^{N-1} \Delta a_i MP + K\Delta a_N P \sum_{i=N}^{N+M-1} (N+M) - K\Delta a_N P \sum_{i=N}^{N+M-1} i \\
&= \sigma_N + KMP \sum_{i=1}^{N-1} \Delta a_i + K\Delta a_N P (N+M)\left[(N+M-1)-(N-1)\right] - K\Delta a_N P \sum_{i=N}^{N+M-1} i
\end{aligned} \tag{21}$$

Expanding the final sum of equation (21)

$$\begin{aligned}
S_1 \equiv \sum_{i=N}^{N+M-1} i &= \sum_{i=1}^{N+M-1} i - \sum_{i=1}^{N-1} i = \frac{1}{2}\left[(N-1+M)(N+M)-(N-1)N\right] \\
&= \frac{1}{2}\left[(N-1)M + MN + M^2\right] \\
&= MN - \frac{1}{2}M + \frac{1}{2}M^2
\end{aligned} \tag{22}$$

Substituting (22) into (21) gives

$$\begin{aligned}
\sigma_{N+M} &= \sigma_N + KMP \sum_{i=1}^{N-1} \Delta a_i + K\Delta a_N P (N+M)M - K\Delta a_N P\left[MN - \frac{1}{2}M + \frac{1}{2}M^2\right] \\
&= \sigma_N + KMP \sum_{i=1}^{N-1} \Delta a_i + \frac{1}{2} K\Delta a_N P\left(M^2 + M\right) \\
&= \sigma_N + KMP \sum_{i=1}^{N} \Delta a_i + \frac{1}{2} KP\Delta a_N \left(M^2 + M\right) - KMP\Delta a_N \\
&= \sigma_N + KMP \sum_{i=1}^{N} \Delta a_i + KP\Delta a_N\left[\frac{1}{2}M(M-1)\right]
\end{aligned} \tag{23}$$

which reduces to equation (18) when $M = 1$. Again defining the auxillary function

$$\begin{aligned}
\delta_N &= MPK \sum_{i=1}^{N} \Delta a_i \\
\delta_{N+M} &= MPK \sum_{i=1}^{N+M} \Delta a_i \\
&= \delta_N + MPK \sum_{i=N+1}^{N+M} \Delta a_N \\
&= \delta_N + M^2 PK\Delta a_N
\end{aligned} \tag{24}$$

The general algorithm may be written as

$$\boxed{\begin{aligned} \delta_1 &= M^2 PK\Delta a_1 \\ \delta_{N+M} &= \delta_N + M^2 PK\Delta a_N \\ \sigma_1 &= 0 \\ \sigma_{N+M} &= \Delta\lambda_N + \delta_N + KP\Delta a_N\left[\tfrac{1}{2}M(M-1)\right] \end{aligned}} \qquad \text{[Algorithm (A)]} \qquad (25)$$

Note that algorithm (A) reduces to equations (20) when $M = 1$, and to equation (13) when $M = 1$ and Δa is a constant..

Variable error optimistic model.

Algorithm. Let the propagation step size be M orbits. Ignoring the change in period, the time after N orbits is $t = t_N = NP$. Use the notation $\sigma_N = \sigma_{\Delta\lambda, da/dt}(t_N)$. Then the error envelope is calculated according to the following algorithm:

$$\boxed{\begin{aligned} K &= \frac{3}{2}\frac{\omega_e}{a} \\ \alpha_1 &= 0 \\ \gamma_1 &= 0 \\ \sigma_1 &= 0 \\ \beta_N &= \tfrac{4}{3}P^2(\Delta a_N)\left(M^2 - \tfrac{3}{8}M + \tfrac{1}{8}\right)^2 \\ \sigma_{N+M} &= \sqrt{\sigma_N^2 + K^2\left[\left(M^2 + 2M\right)\alpha_N + M\gamma_N + \beta_N\right]} \\ \alpha_{N+M} &= \alpha_N + MP^2(\Delta a_N)^2 \\ \gamma_{N+M} &= 2M\alpha_N + \gamma_N + M(M-1)P^2(\Delta a_N)^2 \end{aligned}} \qquad \textbf{[Algorithm B]}$$

Derivation. In this section, the notation σ_N is used to indicate the 1-σ uncertainty in the ground track (i.e., $\sigma_N \equiv \sigma_{\Delta\lambda, da/dt}(t_N)$ where $t = t_N = NP$).

In this model, an error Δa_i is also introduced each orbit i. The error from orbit i propagates according to equation (6). The errors are assumed to add in quadrature. Equation (17) is modified as

$$\sigma_N = K\sqrt{\sum_{i=1}^{N-1}(\Delta a_i)^2(t_N - t_i)^2} \qquad (26)$$

where $K = 3\omega_e/2a$. One orbit later, at $t_{N+1} = t_N + P$,

$$\sigma_{N+1} = K\sqrt{\sum_{i=1}^{N}(\Delta a_i)^2(t_N + P - t_i)^2} \qquad (27)$$

Dividing by K and squaring both sides of the equation,

$$
\begin{aligned}
\left(\sigma_{N+1}/K\right)^2 &= \sum_{i=1}^{N} (\Delta a_i)^2 (t_N + P - t_i)^2 \\
&= \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N + P - t_i)^2 + (\Delta a_N)^2 (t_N + P - t_N)^2 \\
&= \sum_{i=1}^{N-1} (\Delta a_i)^2 \left[(t_N - t_i)^2 + 2P(t_N - t_i) + P^2 \right] + P^2 (\Delta a_N)^2 \\
&= \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N - t_i)^2 + \sum_{i=1}^{N-1} (\Delta a_i)^2 \, 2P(t_N - t_i) + P^2 \sum_{i=1}^{N} (\Delta a_i)^2 \\
&= \left(\sigma_N/K\right)^2 + 2P \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N - t_i) + P^2 \sum_{i=1}^{N} (\Delta a_i)^2
\end{aligned}
\tag{28}
$$

and hence

$$
\sigma_{N+1} = \sqrt{\sigma_N^2 + K^2 (\alpha_{N+1} + \gamma_{N+1})} \tag{29}
$$

where

$$
\alpha_N = P^2 \sum_{i=1}^{N-1} (\Delta a_i)^2, \qquad N > 1 \tag{30}
$$

$$
\gamma_N = 2P \sum_{i=1}^{N-2} (\Delta a_i)^2 (t_{N-1} - t_i), \qquad N > 2 \tag{31}
$$

Recursion relations are also required to efficiently compute the auxillary functions of equations (30) and (31). They are

$$
\alpha_{N+1} = \alpha_N + P^2 (\Delta a_N)^2 \tag{32}
$$

$$
\begin{aligned}
\gamma_{N+1} &= 2P \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N - t_i) \\
&= 2P \sum_{i=1}^{N-2} (\Delta a_i)^2 (t_N - t_{N-1} + t_{N-1} - t_i) + 2P (\Delta a_{N-1})^2 (t_N - t_{N-1}) \\
&= 2P^2 \sum_{i=1}^{N-2} (\Delta a_i)^2 + 2P \sum_{i=1}^{N-2} (\Delta a_i)^2 (t_{N-1} - t_i) + 2P^2 (\Delta a_{N-1})^2 \\
&= \gamma_N + 2\alpha_N
\end{aligned}
\tag{33}
$$

Combining equations (29), (32), and (33) the algorithm may be written as

$$
\boxed{
\begin{aligned}
\alpha_1 &= 0 \\
\alpha_{N+1} &= \alpha_N + P^2 (\Delta a_N)^2 \\
\gamma_1 &= 0 \\
\gamma_{N+1} &= \gamma_N + 2\alpha_N \\
\sigma_1 &= 0 \\
\sigma_{N+1} &= \sqrt{\sigma_N^2 + K^2 (\alpha_{N+1} + \gamma_{N+1})}
\end{aligned}
}
\tag{34}
$$

When the propagation step size is increased to M orbits but the errors are still being added each orbit, equation (34) must be suitably modified. From equation (26), after $N+M$ orbits, the aggregate error function is

$$
\begin{aligned}
\sigma_{N+M}^2 &= K^2 \sum_{i=1}^{N+M-1} (\Delta a_i)^2 (t_{N+M} - t_i)^2 \\
&= K^2 \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_{N+M} - t_N + t_N - t_i)^2 + K^2 \sum_{i=N}^{N+M-1} (\Delta a_i)^2 (t_{N+M} - t_i)^2 \\
&= K^2 \sum_{i=1}^{N-1} (\Delta a_i)^2 (MP + t_N - t_i)^2 + K^2 P^2 \sum_{i=N}^{N+M-1} (\Delta a_i)^2 (N+M-i)^2 \\
&= K^2 M^2 P^2 \sum_{i=1}^{N-1} (\Delta a_i)^2 + K^2 \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N - t_i) \\
&\quad + 2MPK^2 \sum_{i=1}^{N-1} (\Delta a_i)^2 (t_N - t_i)^2 \\
&\quad + K^2 P^2 \sum_{i=N}^{N+M-1} (\Delta a_i)^2 \left[(N+M)^2 - 2i(N+M) + i^2 \right] \\
&= K^2 M^2 \alpha_N + \sigma_N^2 + MK^2 \gamma_{N+1} + MK^2 P^2 (N+M)^2 (\Delta a_N)^2 \\
&\quad - 2K^2 P^2 (\Delta a_N)^2 (N+M) \sum_{i=N}^{N+M-1} i + K^2 P^2 (\Delta a_N)^2 \sum_{i=N}^{N+M-1} i^2 \\
&= \sigma_N^2 + K^2 \left\{ M^2 \alpha_N + M \gamma_{N+1} + P^2 (\Delta a_N)^2 A \right\}
\end{aligned}
\tag{35}
$$

where

$$A = M(N+M)^2 - 2(N+M)S_1 + S_2 \tag{36}$$

S_1, is given by equation (22), and

$$
\begin{aligned}
S_2 \equiv \sum_{i=N}^{N+M-1} i^2 &= \sum_{i=1}^{N+M-1} i^2 - \sum_{i=1}^{N-1} i^2 \\
&= \frac{1}{6}\left[(N+M-1)(N+M)(2N+2M-1) - (N-1)(N)(2N-1) \right] \\
&= \frac{1}{6}\Big[2\left(N^3 + 3N^2 M + 3NM^2 + M^3\right) - 3\left(N^2 + 2MN + M^2\right) + \\
&\qquad N + M - 2N^3 + N^2 + 2N^2 - N \Big] \\
&= N^2 M + NM^2 + \frac{1}{3} M^3 - MN - \frac{1}{2} M^2 + \frac{1}{6} M
\end{aligned}
\tag{37}
$$

The middle term in equation (36) can be expanded by means of equation (22),

$$
\begin{aligned}
2(N+M)S_1 &= 2(N+M)\frac{1}{2}\left(2NM + M^2 - M\right) \\
&= 2N^2 M + 3NM^2 - MN
\end{aligned}
\tag{38}
$$

Substituting equations (37) and (38) into equation (36) gives

$$\begin{aligned} A &= MN^2 + 2M^2N + M^3 - 2MN^2 - 3M^2N + MN \\ &\quad + MN^2 + M^2N + \tfrac{1}{3}M^3 - MN - \tfrac{1}{2}M^2 + \tfrac{1}{6}M \\ &= \tfrac{4}{3}M\left(M^2 - \tfrac{3}{8}M + \tfrac{1}{8}\right) \end{aligned} \tag{39}$$

Combining equations (39) and (35) gives

$$\sigma_{N+M}^2 = \sigma_N^2 + K^2\left[M^2\alpha_N + M\gamma_{N+1} + P^2(\Delta a_N)^2\,\tfrac{4}{3}M\left(M^2 - \tfrac{3}{8}M + \tfrac{1}{8}\right)\right] \tag{40}$$

which, as expected, reduces to equation (29) when M = 1. Substituting equation (33) into equation (40),

$$\begin{aligned} \sigma_{N+M}^2 &= \sigma_N^2 + K^2\left[\left(M^2 + 2M\right)\alpha_N + M\gamma_N + P^2(\Delta a_N)^2\,\tfrac{4}{3}M\left(M^2 - \tfrac{3}{8}M + \tfrac{1}{8}\right)\right] \\ &= \sigma_N^2 + K^2\left[\left(M^2 + 2M\right)\alpha_N + M\gamma_N + \beta_N\right] \end{aligned} \tag{41}$$

where the new function β has been defined as

$$\beta_N = \tfrac{4}{3}P^2(\Delta a_N)^2 M\left(M^2 - \tfrac{3}{8}M + \tfrac{1}{8}\right) \tag{42}$$

M-step recursion relations for the functions α and γ are also required for implementation. Starting with α,

$$\begin{aligned} \alpha_{N+M} &= P^2\sum_{i=1}^{N+M-1}(\Delta a_i)^2 \\ &= P^2\sum_{i=1}^{N-1}(\Delta a_i)^2 + P^2\sum_{i=N}^{N+M-1}(\Delta a_N)^2 \\ &= \alpha_N + MP^2(\Delta a_N)^2 \end{aligned} \tag{43}$$

which satisfies the necessary requirement of reducing to equation (32) when $M = 1$. Similarly,

$$\begin{aligned} \gamma_{N+M} &= 2P\sum_{i=1}^{N+M-2}(\Delta a_i)^2(t_{N+M-1} - t_i) \\ &= 2P\sum_{i=1}^{N-2}(\Delta a_i)^2(t_{N+M-1} - t_i) + 2P(\Delta a_{N-1})^2(t_{N+M-1} - t_{N-1}) + \\ &\qquad 2P\sum_{i=N}^{N+M-2}(\Delta a_N)^2(t_{N+M-1} - t_i) \\ &= S_3 + 2MP^2(\Delta a_{N-1})^2 + S_4 \end{aligned} \tag{44}$$

where S_3 and S_4 represent the two sums in the middle line of equation (44). The first one is

$$
\begin{aligned}
S_3 &= 2P\sum_{i=1}^{N-2}\left(\Delta a_i\right)^2\left(t_{N+M-1}-t_{N-1}+t_{N-1}-t_i\right)\\
&= 2P\sum_{i=1}^{N-2}\left(\Delta a_i\right)^2 MP + 2P\sum_{i=1}^{N-2}\left(\Delta a_i\right)^2\left(t_{N-1}-t_i\right)\\
&= 2MP^2\left[\sum_{i=1}^{N-1}\left(\Delta a_i\right)^2-\left(\Delta a_{N-1}\right)^2\right]+\gamma_N\\
&= 2M\alpha_N+\gamma_N-2MP^2\left(\Delta a_{N-1}\right)^2
\end{aligned}
\tag{45}
$$

Referring to equation (22), S_4 can be simplified as

$$
\begin{aligned}
S_4 &= 2P\left(\Delta a_N\right)^2\sum_{i=N}^{N+M-2}P(N+M-1-i)\\
&= 2P^2\left(\Delta a_N\right)^2\left[\sum_{i=N}^{N+M-2}(N+M-1)-\sum_{i=N}^{N+M-2}i\right]\\
&= 2P^2\left(\Delta a_N\right)^2\left\{(M-1)(N+M-1)-\left[S_1-(N+M-1)\right]\right\}\\
&= 2P^2\left(\Delta a_N\right)^2\left\{(M-1)\left[N+(M-1)\right]-\left[MN-\tfrac{1}{2}M+\tfrac{1}{2}M^2-(N+M-1)\right]\right\}\\
&= 2P^2\left(\Delta a_N\right)^2\left\{\tfrac{1}{2}M(M-1)\right\}\\
&= M(M-1)P^2\left(\Delta a_N\right)^2
\end{aligned}
\tag{46}
$$

Substituting equations (45) and (46) into (44),

$$
\begin{aligned}
\gamma_{N+M} &= S_3+2MP^2\left(\Delta a_{N-1}\right)^2+S_4\\
&= \left[2M\alpha_N+\gamma_N-2MP^2\left(\Delta a_{N-1}\right)^2\right]+2MP^2\left(\Delta a_{N-1}\right)^2+M(M-1)P^2\left(\Delta a_N\right)^2\\
&= 2M\alpha_N+\gamma_N+M(M-1)P^2\left(\Delta a_N\right)^2
\end{aligned}
\tag{47}
$$

Which reduces to equation (33) when $M = 1$.

Equations (41), (42), (43) and (47) may be combined to give algorithm B,

$$
\boxed{
\begin{aligned}
K &= \frac{3}{2}\frac{\omega_e}{a}\\
\alpha_1 &= 0\\
\gamma_1 &= 0\\
\sigma_1 &= 0\\
\beta_N &= \tfrac{4}{3}P^2\left(\Delta a_N\right)\left(M^2-\tfrac{3}{8}M+\tfrac{1}{8}\right)^2\\
\sigma_{N+M} &= \sqrt{\sigma_N^2+K^2\left[\left(M^2+2M\right)\alpha_N+M\gamma_N+\beta_N\right]}\\
\alpha_{N+M} &= \alpha_N+MP^2\left(\Delta a_N\right)^2\\
\gamma_{N+M} &= 2M\alpha_N+\gamma_N+M(M-1)P^2\left(\Delta a_N\right)^2
\end{aligned}
}
\tag{48}
$$

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